Theorem 3.1. When Condition 1.1 is fulfilled, the first player's position strategy U exists, which guarantees the estimate

$$\min_t \rho(\{x[t]\}_m, M) \leqslant \alpha$$
$$t_0 \leqslant t \leqslant \vartheta, \quad \alpha = \max\{0, \varepsilon(t_0, x_0)\}$$

for any motion $x[t] = x[t; t_0, x_0, U]$.

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ON A GENERALIZATION OF THE THEORY OF ERROR ACCUMULATION

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We pose the problem of error accumulation in linear systems on a finite time interval under three constraints of the perturbation function and its lower derivatives. We have shown that the largest error in the system is realized in the class of piecewise-quadratic functions possessing certain limit properties on the set of switching points of the system's impulse transient response and of the maximizing external influence. Schemes are obtained for the effective solution of the problem, based on a combination of Bellman's optimality principle and of analytic information on the extremal properties of the external influences. The present paper is a development of [1 − 3].

1. Statement of the problem. Let the error in the kth system coordinate x_k $(k = 1, \ldots, l)$, caused by a perturbing action $f(t)$, be the solution of the differential equation

$$\frac{d^n x_k}{dt^n} = \sum_{i=0}^{n-1} A_i(t) \frac{d^i(x_k)}{dt^i} + f(t) \tag{1.1}$$

in which the coefficients $A_i(t)$ $(i = 0, \ldots, n-1)$ are continuous functions of time t on the interval $[0, T_0]$ and $f(t)$ belongs to a set M of functions satisfying the following constraints:

$$|f(t)| \leqslant m_0, \qquad |f'(t)| \leqslant m_1 \qquad (1.2)$$
$$|f'(t) - f'(s)| \leqslant m_2 |t - s|$$

The solution of Eq. (1.1) can be written as

$$x_k(f, T) = \int_0^T k(T, t) f(t)\, dt, \qquad T \in J = (0, T_0] \qquad (1.3)$$

Here $k(T, t)$ is the system's impulse transient response relative to perturbation $f(t)$. The problem is to seek the quantity

$$\max_{T \in J} \max_{f \in M} x_k(f, T) \qquad (1.4)$$

When $T = T_0$ problem (1.2) – (1.4) degenerates to the know problem of error accumulation at a fixed instant T_0, considered in [4 – 8] with one constraint on $f(t)$, in [9] with two constraints (on the function and on the first derivative), and in [1 – 3] under three constraints, respectively. In linear systems with constant parameters with one constraint on $f(t)$ ($|f(t)| \leqslant m_0$) the maximum error in the interval $[0, T_0]$ is realized at the extreme point T_0. In the remaining cases this maximum can be reached at an intermediate point of the interval $[0, T_0]$.

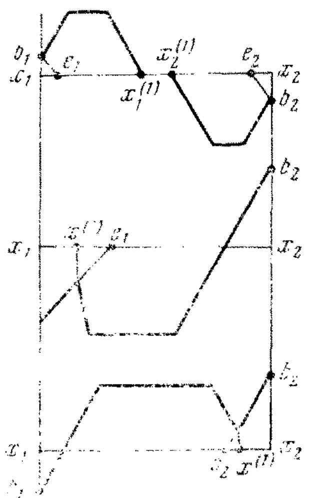

Fig. 1

2. Error accumulation at a fixed instant T.

We consider an auxiliary problem on the limit values of the functional

$$I(F; x_3) = \int_{x_1}^{x_3} F(x)\, dx \qquad (x_1 \leqslant x_3 \leqslant x_2) \qquad (2.1)$$

in the class R of functions $F(x)$ subject to the following requirements:

$$|F(x)| \leqslant m_1, \ |F(x) - F(s)| \leqslant m_2 |x - s|;$$
$$F(x_i) = b_i \ (i = 1, 2) \quad I(F; x_2) = C,$$
$$C_1 \leqslant I(F; x_3) \leqslant C_2, \quad (C_1 \leqslant 0, C_2 \geqslant 0)$$

We delineate a family $X \subset R$ of continuous piecewise-linear functions

$$\Phi_i(x) = \Phi_i(x; x_1^{(i)} x_2^{(i)}), \quad (i = 1, 2, \ x_1 \leqslant x_1^{(i)} \leqslant x_2^{(i)} \leqslant x_2, \ \ x_1^{(i)} = x_2^{(i)})$$
$$\text{for } x_1^{(i)} \leqslant x_1 + \frac{|b_1|}{m_2} \quad \text{and for} \quad x_2^{(i)} \geqslant x - \frac{|b_2|}{m_2}$$

The function $\Phi_1(x)$ has the form (Fig. 1)

$$\Phi_1(x) = \begin{cases} \min\{b_1 + m_2(x - x_1);\ m_1;\ m_2(x_1^{(1)} - x)\} & (x_1 \leqslant x \leqslant x_1^{(1)}) \\ 0 & (x_1^{(1)} \leqslant x \leqslant x_2^{(1)}) \\ \max\{m_2(x_2^{(1)} - x);\ -m_1;\ b_2 - m_2(x_2 - x)\} & (x_2^{(1)} \leqslant x \leqslant x_2) \end{cases} \qquad (2.2)$$

$$\Phi_1(x) = \begin{cases} b_1 - \operatorname{sign}(b_1)\, m_2 (x - x_1) & (x_1 \leqslant x \leqslant x_1^{(1)}) \\ \max\{\Phi_1(x^{(1)}) - m_2(x - x^{(1)});\ -m_1;\ \ b_2 - m_2(x_2 - x)\} \\ (x^{(1)} \leqslant x \leqslant x_2) \end{cases}$$

for $x_1^{(1)} < x_1 + b_1/m_2$

$$\Phi_1(x) = \begin{cases} \min\{b_1 + m_2(x - x_1);\ m_1;\ b_2 - \operatorname{sign}(b_2)\, m_2(x_2 - x') + m_2(x^{(1)} - x)\} \\ \qquad (x_1 \leqslant x \leqslant x^{(1)}) \\ b_2 - \operatorname{sign}(b_2)\, m_2(x_2 - x) \qquad (x^{(1)} \leqslant x \leqslant x_2) \end{cases}$$

for $x_1^{(1)} > x_2 - |b_2|/m_2$

The function $\Phi_2(x) = \Phi_2(x;\ x_1^{(2)},\ x_2^{(2)})$ is defined analogously. The following lemma is valid (the proof was presented in [2]).

Lemma 1. If set R is not empty, then the largest value of functional (2.1) for any fixed value of x is reached on one and the same piecewise-linear function $\Phi_1(x;\ u^{(1)},\ v^{(1)}) \in X$ possessing the property

$$\int_{x_1}^{u^{(1)}} \Phi_1(x, u^{(1)}, v^{(1)})\, dx = \sup_{x^{(1)}} \int_{x^1}^{x_1^{(1)}} \Phi_1(x; x_1^{(1)}, v^{(1)})\, dx$$

An analogous assertion is valid relative to the function $\Phi_2(x,\ u^{(2)},\ v^{(2)}) \in X$, on which the smallest value of the functional is reached.

Let $L[x_1,\ x_2]$ be a family of functions $f(x) \in M$, given on the segment $[x_1,\ x_2]$ and satisfying the conditions

$$f(x_i) = a_i, \qquad f'(x_i) = b_i \qquad (i = 1,2) \tag{2.3}$$

The set of derivatives of $f(x) \in L$ belongs to R for certain values $C = a_2 - a_1$, $C_1 = -m_0 - a_1$ and $C_2 = m_0 - a_1$. If L is nonempty, then by virtue of Lemma 1, applied to the derivatives of $f(x) \in L$, we can find functions $f_b(x) \in L$ and $f_H(x) \in L$ realizing the relations

$$f_b(x) = \sup_{f \in L} f(x), \qquad f_H(x) = \inf_{f \in L} f(x)$$

for any $x \in (x_1,\ x_2)$. The functions $f_b(x)$ and $f_H(x)$ are called the upper and the lower function of set L, respectively.

Further, let $L_1[x_1,\ x_2]$ and $L_2(x_1,\ x_2]$ be families of functions $f(x) \in M$ satisfying the first condition in (2.3) for one and the same values a_i and the second condition in (2.3) for the values $b_i = b_i'$ for $f(x) \in L_1$ and $b_i = b_i''$ for $f(x) \in L_2$. According to Lemma 1, each of the sets $L_j[x_1,\ x_2]$ $(j = 1,\ 2)$, if it is not empty, contains its own upper $f_{bj}(x)$ and lower $f_{Hj}(x)$ functions.

Lemma 2. Let (a) each of the sets $L_j[x_1,\ x_2]$ $(j = 1,\ 2)$ be nonempty: (b) the derivatives $f'(x_i)$ $(i = 1,\ 2)$ of the functions $f(x) \in L_1$ and $f(x) \in L_2$ satisfy the conditions: $b_1 \geqslant b''_1$ and $b_2' \leqslant b_2$. Then for any $x \in [x_1,\ x_2]$ we have

$$f_{b1}(x) \geqslant f_{b2}(x), \qquad f_{H1}(x) \leqslant f_{H2}(x) \tag{2.4}$$

Proof. By the definition of an upper function the derivative $f_{bj}'(x)$ $(j = 1,\ 2)$ is the piecewise-linear function $\Phi_1(x;\ u_j;\ v_j)$.

1°. Let $u_1 < v_1$. According to Lemma 1, when $u_1 < v_1$ the upper function $f_{b_1}(x) = m_0\ (u_1 \leqslant x \leqslant v_1)$. But then, by the lemma's hypotheses,

$$u_1 \leqslant u_2 \leqslant v_2 \leqslant v_1$$
$$f_{b_1}'(x) = \geqslant f_{b_2}'(x) \qquad (x_1 \leqslant x \leqslant y_1,\ v_2 \leqslant x \leqslant y_2) \tag{2.5}$$
$$f_{b_1}'(x) = \leqslant f_{b_2}'(x) \qquad (y_1 \leqslant x \leqslant u_2,\ y_2 \leqslant x \leqslant x_2)$$

Here y_1 (y_2) is a boundary of the segment (y_1, u_1) (respectively, (v_1, y_2)) on which $f_{b_1}''(x) = -m_2$. By virtue of (2.5) the first relation in (2.4) is automatically fulfilled on the segments $[x_1, y_1]$ and $[y_2, x_2]$.

Passing to the segment $[y_1, u_2]$, we assume that the function $f_{b_1}(x_a) < f_{b_2}(x_a)$ at some point x_a $(y_1 < x_a < y_2)$. But then there is the inevitable contradiction

$$f_{b2}(x) \notin L_2[x_1, x_2] \tag{2.6}$$

since

$$f_{b_2}(u_1) = f_{b_2}(x_a) + \int_{x_a}^{u_1} f_{b_2}'(x)\,dx > f_{b_1}(x_a) + \int_{x_a}^{u_1} f_{b_1}'(x)\,dx = m_0$$

In exactly the same way the nonfulfillment of condition (2.4) at any point of the segment $[v_1, y_2]$ implies the analogous inequality: $f_{b_2}(v_2) > m_0$.

2°. Let $u_1 = v_1$. Then also $u_2 = v_2$, since otherwise by virtue of inequality (2.6) when $u_2 < u_1$ or by virtue of the relation

$$f_{b_2}(x_2) = f_{b_2}(v_2) + \int_{v_2}^{x_2} f_{b_2}'(x)\,dx > f_{b_1}(v_2) + \int_{v_2}^{x_2} f_{b_1}'(x)\,dx = a_2 \tag{2.7}$$

when $u_2 \geqslant u_1$, there is the inevitable contradiction: $f_{b_2}(x) \notin L_2$.

So long as $u_j = v_j$ $(j = 1, 2)$, this case differs from the general case considered in 1° only in the existence of a subsegment (y_1', y_2') of segment (x_1, x_2), on which $f_{b_2}''(x) = -m_2$ and either $f_{b_1}'(x) \geqslant f_{b_2}'(x)$ or $f_{b_1}'(x) \leqslant f_{b_2}'(x)$. In the first case $(f_{b_1}'(x) \geqslant f_{b_2}'(x)$ $(y_1' \leqslant x \leqslant y_2'))$ the first relation in (2.4) is fulfilled automatically on the segment $[y_1', y_2']$. In the second case we arrive at (2.7) by admitting the existence of a point x_c $(y_1' < x_c < y_2')$, where $f_{b_1}(x_c) < f_{b_2}(x_c)$.

The proof is analogous for the lower functions. The lemma is proved.

Passing to the basic problem, we examine a theorem on the largest value of functional (1.3) on set M for a fixed T.

Theorem 1. For a fixed T relation (1.4) is achieved on a set E of piecewise-quadratic functions $f_0(t) \in M$ possessing the following properties:

a) the functions $f_0(t)$ have one and the same second derivative;

b) each of the functions $f_0(t)$ is an upper (respectively, a lower) function of some set $L[t_s, t_{s+1}]$ in each interval $[t_s, t_{s+1}]$ $(s = 0, 1, \ldots, n-1)$, where $k(T, t) \geqslant 0$ $(k(T, t) \leqslant 0)$.

Proof. 1°. Set M is relatively compact by virtue of the equicontinuity and uniform boundedness of the functions $f(t) \in M$ (see (1.2)). Furthermore, M is closed (by virtue of the nonstrict inequalities (1.2)). Consequently, the largest value of functional $I(f)$ is reached on some function $f_m \in M$. Let us assume, to the contrary, that in some interval $(t_r, t_{r+1}]$ the maximizing function is not an upper function when $k(T, t) > 0$ (not a lower function when $k(T, t) \leqslant 0$) of the set

$$L_r = \{f : f \in M; f(t_s) = f_m(t_s), f'(t_s) = f'_m(t_s)\ (s = r, r+1)\}$$

Then, after replacing $f_m(t)$ in the interval $[t_r, t_{r+1}]$ by the upper function when $k(T, t) \geqslant 0$ (by the lower function when $k(T, t) \leqslant 0$) of L_r, the value of the functional is increased.

2°. To prove the uniqueness of the second derivative $f_m''(t)$, we make use of the condition, following from (1.2), that set M is convex. In this case the function

$$f_\lambda(t) = \lambda f_1(t) + (1-\lambda) f_2(t)$$

will belong to M for any λ $(0 \leqslant \lambda \leqslant 1)$ provided $f_1(t)$ and $f_2(t)$ belong to M. At the same time, if $f_1(t)$ and $f_2(t)$ are maximizing functions, then, obviously, $f_\lambda(t)$ $(0 \leqslant \lambda \leqslant 1)$ also is a maximizing function. According to (2.2) the second derivative of the maximizing function $f_0(t) \in M$ can take only the values m_2, $-m_2$ and 0. The second derivative $f_\lambda''(t)$ satisfies this requirement in the single case when $f_1''(t) \equiv f_2''(t)$. In fact, having differentiated $f_\lambda(t)$ twice with respect to t and assuming $f''_1(t) \not\equiv f_2''(t)$, after the substitution of the possible values of $f_1''(t)$ and $f_2''(t)$, equal to m_2, $-m_2$ or 0, we arrive at the contradiction $f_\lambda''(t) = \pm \lambda m_2$ for $0 < \lambda < 1$. The theorem is proved.

3. Error accumulation on a finite time interval. Let $k(T,t) = \bar{k}(t)$, t_s $(s = 1, \ldots, n-1)$ be switching points of kernel $\bar{k}(t)$; $t_0 = 0$, $t_n = T$; $\bar{k}(t) \geqslant 0$ $(t_s \leqslant t \leqslant t_{s+1},\ s = 0, 2, \ldots)$.

Theorem 2. Relation (1.4) is achieved on the product of set E and of a set A consisting of the switching points of the kernel $\bar{k}(t)$ of the functional, of the point T, and of the switching points θ_s of the function $f_0(t) \in E$, belonging to those intervals $[t_s, t_{s+1}]$ $(t_s < \theta_s < t_{s+1})$ of sign-constancy of the kernel, on which the conditions

$$(-1)^s f_0(t_s) > 0, \qquad (-1)^s f_0'(t_s) < 0 \tag{3.1}$$

are fulfilled.

Proof. 1°. Functional $I(f, T)$ is given on the set product $M \times J$. The sets M and J are compact in the corresponding metric spaces $C_{[0,T]}$ and R_1. Consequently, their product is compact, and relation (1.4) is achieved at some point T of the interval $[t_r, t_{r+1}]$ of sign-constancy of the kernel by some function $f_m(t) \in U$. Here $(s = r,\ r+1)$

$$U = \{f : f \in M;\ f(t_s) = a_s,\ f'(t_s) = b_s\}. \tag{3.2}$$

Relation (1.4) can be written as

$$\max_{T \in J} \max_{f \in M} I(f, T) = \int_0^{t_r} \bar{k}(t) f_m(t)\, dt + \max_{t_r \leqslant T \leqslant t_{r+1}} \int_{t_r}^{T} \bar{k}(t) f_m(t)\, dt \tag{3.3}$$

According to Theorem 1 and to (3.2), the first functional in (3.3) reaches its largest value on a piecewise-quadratic function $f^* \in E \cap U$. The maximum over f of the second functional is realized, independently of the value T $(t_r \leqslant T \leqslant t_{r+1})$, by a function $f^{**}(t) \in E \cap U$, whose derivative has the form

$$\frac{d}{dt} f^{**}(t) = \begin{cases} \min\{b_r + m_2(t - t_r);\ m_1;\ m_2(t'_{r+1} - t)\} & (t_r \leqslant t \leqslant t_{r+1}) \\ 0 & (t > t'_{r+1}) \end{cases} \tag{3.4}$$

Here we have chosen t'_{r+1} from the condition $f^{**}(t'_{r+1}) = m_0$ and considered the case when $\bar{k}(t) \geqslant 0$ $(t_r \leqslant t \leqslant t_{r+1})$. In fact, according to Lemma 1, $f^{**}(t)$ is the upper function of the set $U[t_r, t_{r+1}]$. Consequently, the maximum of functional $I(f, T)$ over f for any T is realized by some function $f_0(t) \in E \cap U$, equal to $f^*(t)$ in the interval $[0, t_r]$ and to $f^{**}(t)$ in $[t_r, t_{r+1}]$.

2°. The second integral in (3.3) can reach a maximum value at an intermediate point θ_r of interval $[t_r, t_{r+1}]$ if θ_r is the point at which the function $f^{**}(t)$ changes sign from positive to negative, because then $f^{**}(t) \geqslant 0$ $(t_r \leqslant t \leqslant \theta_r)$. This can happen when conditions (3.1) are fulfilled. In the remaining cases the maximizing function either is sign-constant or can have a single point inside interval $[t_r, t_{r+1}]$ at which it changes from negative to positive, and relation (1.4) is achieved at one of the extreme points of the interval $[t_r, t_{r+1}]$.

The proof for the case $\overline{k}(t) \leqslant 0$ $(t_r \leqslant t \leqslant t_{r+1})$ is carried out analogously. The theorem is proved.

According to Theorem 1 problem (1.2) – (1.4) can be replaced by the finite-dimensional problem of maximizing the function

$$I(f) = \int_0^T \overline{k}(t) f_0(t; a_0, \ldots, a_n;\ b_0, \ldots, b_n)\, dt$$

of $2(n+1)$ variables: $f_0(t_s) = a_s$ and $f_0'(t_s) = b_s$ $(s = 0, 1, \ldots, n)$. If the number of intervals of sign-constancy of the impulse transient response $\overline{k}(t)$ is small, the maximum errors can be found by the usual methods of differential calculus. Bellman's dynamic programing method proves to be the most effective method in the general case.

We consider the functional

$$I_k(a_k, b_k) = \max_{\substack{a_0, \ldots, a_{k-1} \\ b_0, \ldots, b_{k-1}}} \int_0^{t_k} k(T, t) f_0(t; a_0; \ldots, a_{k-1}, b_0, \ldots, b_{k-1}; a_k, b_k)\, dt$$

given on a set of functions $f_0 \in M$ for which the value at point t_k and the value of the derivative at point t_k are fixed: $f_0(t_k) = a_k$, $f_0'(t_k) = b_k$. Bellman's functional equation has the form

$$I_{k+1}(a_{k+1}, b_{k+1}) = \max_{a_k, b_k} \left[I_k(a_k, b_k) + \int_{t_k}^{t_{k+1}} \overline{k}(T, t) f_0(t; a_k, b_k; a_{k+1}, b_{k+1})\, dt \right] \quad (3.5)$$

Its solution on an electronic digital computer can be effected by the following scheme.

Stage 1. Determination of all possible combinations of values of a_k, b_k, a_{k+1}, b_{k+1} $(k = 1, \ldots, n)$, satisfying the condition that the corresponding sets $L[t_k, t_{k+1}]$. are nonempty.

Stage 2. The sequential solution of functional Eqs. (3.5) from $k = 1$ to $k = n$, including:

a) determination of the upper functions, for $k = 0, 2, \ldots$, and of the lower functions, for $k = 1, 3, \ldots$, of the corresponding sets $L[t_k, t_{k+1}]$;

b) computation of functionals $I_k(a_k, b_k)$ by the usual procedure for solving Bellman's functional difference equations;

c) determination of the quantity $\max I_n(a_n, b_n)$.

The functional Eqs. (3.5) are two-dimensional, which causes specific computational difficulties for an effective solution. By using certain properties of optimal controls at the switching points of the functional's kernel, we can substantially simplify the computational scheme.

We consider the set $N_{k+1} = N_{k+1}(a_k^*, b_{k+1}^*, a_{k+1}^*)$ of functions $f_0 \in E$ with the fixed values: $f_0(t_k) = a_k^*$, $f_0'(t_{k+1}) = b_{k+1}^*$ and $f_0(t_{k+1}) = a_{k+1}^*$. By virtue of

Theorem 1 the quantity

$$\max_{\substack{a_0,\ldots,a_{k-1}\\ b_0,\ldots,b_{k-1}}} \int_0^{t_{k+1}} k(T,t)\, f_0(t;a_0,\ldots,a_{k-1},b_0,\ldots,b_{k-1};\ a_k^*\ ,b_k,a_{k+1}^*,b_{k+1}^*)\,dt \tag{3.6}$$

is achieved by some function $f_0(t)$ at some value $b_k \in B_k$, where B_k is the set of values of the derivative $f_0'(t_k)$ $(f_0 \in N_{k+1})$. Let $B_k' \subset B_k$ be a subset of values of the derivative $f_0'(t_k)$, for which $(-1)^k f_0''(t_k - 0) = -m_2$ or 0. The following theorem is valid.

Theorem 3. If set N_{k+1} is not empty, then a solution exists of functional Eq.(3.5) for the values $f_0(t_k) = a_k^*$, $f_0'(t_{k+1}) = b_{k+1}^*$ and $f_0(t_{k+1}) = a_{k+1}^*$, realizable by some function $f_0^* \in N_{k+1}$ for which the derivative at point t_k either belongs to B_k' or is the upper boundary for $k = 0, 2, \ldots$ (respectively, the lower boundary, for $k = 1, 3, \ldots$) of set B_k.

Proof. Let $\bar{k}(t) \geqslant 0$ $(t_{k-1} \leqslant t \leqslant t_k)$; the values a_k, a_{k+1} and b_{k+1} are chosen such that N_{k+1} is not empty; correspondingly, the set B_k is not empty; further, let $b^* = \min b_k (b_k \in B_k)$. Consider the functional equation

$$I_{k+1}'(b) = \max_{a_{k-1},\, b_{k-1}} J(a_{k-1},b_{k-1},b_k) = \max_{a_{k-1},\, b_{k-1}} [I_{k-1}(a_{k-1},b_{k-1}) + \int_{t_{k-1}}^{t_{k+1}} \bar{k}(t)\, f_0(t;a_{k-1},b_{k-1},b_k)\,dt] \quad (f_0 \in N_{k+1}) \tag{3.7}$$

Obviously,

$$I_{k+1}^* = \max_{\substack{a_0,\ldots,a_{k-1}\\ b_0,\ldots,b_k}} \int_0^{t_{k+1}} k(t)\, f_0(t)\,dt = \max_{b\in B_k} I_{k+1}'(b)$$

Let N_{k+1}' be the set of functions $f_0 \in N_{k+1}$ by which the quantity I_{k+1}' is achieved for all $b_k \in B_k$. Arguing to the contrary, assume that I_{k+1}^* is realized by some function $f_{01} \in N_{k+1}'$ satisfying the conditions

$$f_{01}''(t_k) = m_2, \qquad f_{01}'(t_k) = b' > b^* \tag{3.8}$$

According to (2.2) and (3.8) we can form the function $f_{02}(t) \in N_{k+1}$, satisfying the following requirements:

$$\begin{aligned} &f_{02}(t) \equiv f_{01}(t) \qquad (0 \leqslant t \leqslant t_{k+1}) \\ &f_{02}'(t_k) = b_* < b' \\ &f_{02}''(t_k) = -m_2 \qquad \text{or } 0\,(b_* > b^*) \end{aligned} \tag{3.9}$$

From (3.9) and Theorem 1 it follows that the function $f_{02}(t) \in N_{k+1}$ is a solution of functional Eq. (3.7) for some value $b = b_*$. Applying Lemma 2 to the functions $f_{01}(t)$ and $f_{02}(t)$ on every interval $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$, the functions differing between themselves in the value of the derivative at point t_k, we obtain

$$\begin{aligned} &f_{02}(t) \geqslant f_{01}(t) \qquad (t_{k-1} \leqslant t \leqslant t_k) \\ &f_{02}(t) \leqslant f_{01}(t) \qquad (t_k \leqslant t \leqslant t_{k+1}) \end{aligned} \tag{3.10}$$

Then, as a result of replacing the function $f_{01}(t)$ in (3.7) by $f_{02}(t)$, the first term on the right-hand side remains unchanged, while, by virtue of (3. 9), the second term increases by the amount

$$\int_{t_{k-1}}^{t_{k+1}} \bar{k}(t)\,[f_{02}(t) - f_{01}(t)]\,dt > 0$$

which leads to the contradiction

$$J\ (a_{k-1},\ b_{k-1},\ b') < J\ (a_{k-1},\ b_{k-1},\ b_*)$$

The proof is analogous for the case $\bar{k}\ (t) \leqslant 0\ \ (t_{k-1} \leqslant t \leqslant t_k)$. The theorem is proved.

We can considerably simplify the solution of functional Eqs. (3. 5) by using Theorem 3, because in this case the straightforward sorting through of solutions for all values of a_k and b_k is replaced by a sorting through of solutions only for the values $b_k \in B_k'$ and for one of the boundary values b_k^* (the minimum one for odd values of k and the maximum one for even values).

By virtue of Theorem 2 an effective solution of problem (1. 2) – (1. 4) can be reduced to solving the same functional Eqs. (3. 5) as for problem (1. 2) – (1. 4) when $T = T_0$, with the sole difference that at each stage of solving Eqs. (3. 5) (for each value of k) it is necessary to determine the quantity

$$I_k^* = \max_{a_k,\ b_k} I_k(a_k, b_k) \qquad (k = 1, \ldots, n-1) \tag{3.11}$$

and to compute the integral

$$I_k' = \int_0^{\theta_k} \bar{k}(t)\, f_0^*(t)\, dt \qquad (f_0^*(\theta_k) = 0,\ t_k < \theta_k < t_{k+1}) \tag{3.12}$$

over functions $f^*_0(t)$ which satisfy the hypotheses of Theorem 2. The largest error in the system in the interval $[0, T_0]$ is determined by comparing the quantities I_k^*, I_k' and I_n $(k = 0,\ 1,\ \ldots,\ n-1)$ found from (3.11) and (3.12).

Example. The impulse transient response is given in the form

$$k\ (t) = 10e^{-0.6t} \sin t$$

It is known that the perturbation $f\ (t)$ belongs to some class M with constants: $m_0 = 1$, $m_1 = 0.715$, $m_2 = 0.35$. We are required to determine

$$\max_{0 \leqslant T' \leqslant 2\pi} \max_{f \in M} x\,(T', f) = \max_{0 \leqslant T' \leqslant 2\pi} \max_{f \in M} \int_0^{T} k\,(t)\, f\,(T' - t)\, dt \tag{3.13}$$

Solution. We solve the functional Eqs. (3. 5) successively for the intervals $[0, \pi]$ and $[\pi, 2\pi]$. We find

$$\max_{f \in M} x\,(\pi) = 8.4, \qquad f_m(\pi) = 1, \qquad f_m'(\pi) = 0 \tag{3.14}$$

$$f_m\ (t) = 1 \qquad (0 \leqslant t \leqslant \pi)$$

$$\max_{f \in M} x\,(2\pi) = 7.8, \qquad f_{m1}(\pi) = 0.42, \qquad f_{m1}'(\pi) = -0.637$$

$$f_{m_1}(t) = 1\ (0 \leqslant t \leqslant \tau), \qquad f_{m_1}(2\pi) = -0.23, \qquad f_{m1}'\ (2\pi) = -0.715$$

$$f_m'\ (t) = \begin{cases} 0 & (0 \leqslant t \leqslant \tau) \\ \max\{-m_2\ (t - \tau);\ -m_1\} & (\tau \leqslant t \leqslant 2\pi) \end{cases}$$

Thus, relation (3.13) is realized by function (3.14)

$$\max_{0 \leqslant T' \leqslant 2\pi} \max_{f \in M} x\,(T', f) = x\,(\pi, f_m) = 8.4$$

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USE OF THE VARIATIONAL EQUATION IN THE STUDY OF POLARIZABLE AND MAGNETIZABLE CONTINUOUS MEDIA

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A model of a continuum is constructed, using the variational equation suggested in [1, 2] which makes it possible to obtain models of continua using a minimum number of unified physical hypotheses. In the present paper the variational equation is used to obtain a system of equations defining the macroscopic motion of a continuum with polarization and magnetization effects taken into account, within the framework of the special relativity theory. Use of the four-dimensional space-time and special relativity theory is required in order to match theories of electromagnetism and mechanics. We investigate some of the consequences of two possible decompositions of the total energy-momentum tensor of the electromagnetic field and the continuum into the continuum energy-momentum tensor and the electromagnetic field energy-momentum tensor according to Minkowski and to Abraham, respectively. When moment stresses and external mass moments are absent in the medium, we assume the symmetry of the total energy-momentum tensor of electromagnetic field and medium (this is equivalent to the absence or constancy of the combined electromagnetic